# MATH 141: Midterm 1

Name: key

Directions:

* Show your thought process (commonly said as "show your work") when solving each problem for full credit.
* If you do not know how to solve a problem, try your best and/or explain in English what you would do.
* Good luck!

| Problem | Score | Points |
|---|---|---|
| 1 | | 10 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| | | **90** |

work

or conceptual understanding which keeps my work correct

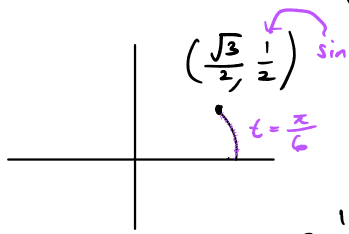
common mistakes to avoid.

1. If

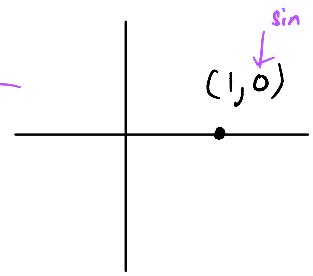
$$f(x) = x^2 - x \qquad g(x) = 3x^2 - x + 1 \qquad h(x) = \sin(x) \qquad j(x) = 2^x$$

Evaluate, expand, and/or simplify the following:

(a) $h\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \boxed{\frac{1}{2}}$




(b) $j(1) \cdot h(0) = 2^1 \cdot \sin(0)$




$= 2 \cdot 0$

$= \boxed{0}$

(c) $\underbrace{f(x)}_{\text{two term}} \cdot \underbrace{g(x)}_{\text{three term.}}$

Don't forget parenthesis when multiplying into ≥2 terms!

$$f(x) \cdot g(x) = (x^2 - x)(3x^2 - x + 1) \overset{\text{dist}}{\underset{\text{law}}{=}} x^2(3x^2 - x + 1) + (-x)(3x^2 - x + 1)$$

$$\overset{\text{dist law}}{=} 3x^4 \boxed{-x^3} + \boxed{x^2} \boxed{-3x^3} + \boxed{x^2} - x$$

$$= \boxed{3x^4 - 4x^3 + 2x^2 - x}$$

(d) $f(x + h) - f(x)$

Since $f(x) = x^2 - x$

look! x+h replaces the "x" visually! Now do it!

$$f(\boxed{x+h}) - f(x) = \underbrace{(x+h)^2 - (x+h)}_{f(x+h)} - \underbrace{(x^2 - x)}_{f(x)}$$

Common mistake: forget the parenthesis!

$$\overset{\text{expand,}}{\underset{\text{dist law}}{=}} x^2 + 2xh + h^2 - x - h - x^2 + x$$

$$= 2xh + h^2 - h$$

$$\overset{\text{GCF}}{=} \boxed{h(2x + h - 1)^2}$$

2. Short answer questions:

(a) When you are given the directive "Simplify this expression.." what does the word **simplify** mean??

Simplify means to try to break down an expression into global factors. it also means combine like terms. it also means simplify fractional expressions so there is only one fraction.

$\boxed{ex}$ $(x+h)^2 - x^2$ is not simplified b/c $(x+h)^2$ generates a like term, $x^2$.

Therefore, you must always consider expanding to get rid of like terms.

(b) True or false: We can simplify

$$\frac{3(x-2)^2(x+3)-4(x+2)(x-3)^2}{5x(x-3)^2(x-2)-4(x+3)}$$

by crossing out the $x+3$.

No. $(x+3)$ is only a local factor in one term context in the numerator and denominator.

(c) If a function is differentiable, is it continuous?

Yes. This is the theorem.

(d) If $F(x) = \sin^3(x^2)$ find three functions $f, g, h$ where $f \circ g \circ h = F$.

$$\boxed{\begin{aligned} f(x) &= x^3 \\ g(x) &= \sin(x) \\ h(x) &= x^2 \end{aligned}}$$

Verifying:

deal w/ $h(x)$ first, we know what it is.

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

$$= f(g(x^2))$$

$x^2$ takes the place of $x$ in $g(x)$

$$= f(\sin(x^2))$$

$\sin(x^2)$ replaces $x$ in $f(x)$

$$= (\sin(x^2))^3 = \sin^3(x^2) = F(x) \checkmark$$

Common mistake #1: plugging in 2 into $-x^2+1$ is $-2^2+1 = (-1)\cdot 2^2 + 1 = -4+1 = \boxed{-3}$ negative law #1.

3. Suppose

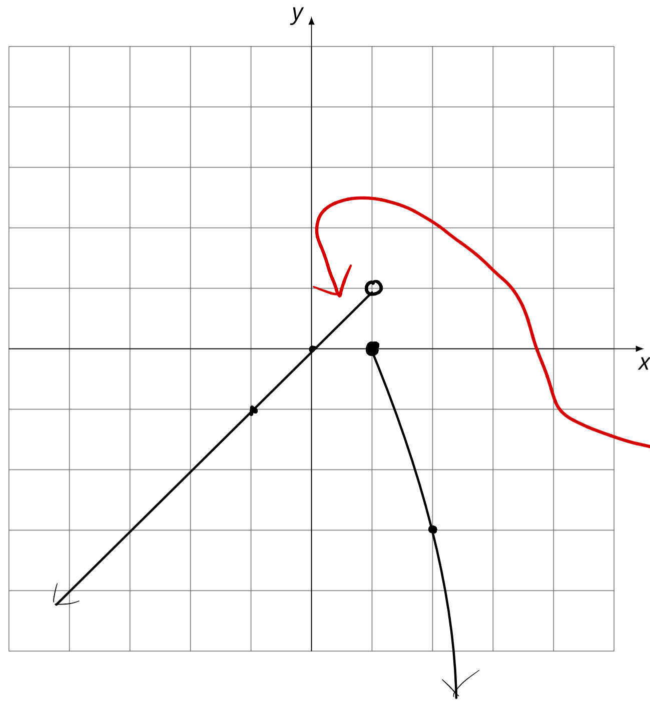
$$f(x) = \begin{cases} x & \boxed{x < 1} \\ \boxed{-x^2+1} & x \geq 1 \end{cases}$$

Common mistake #2: $x<1$ is

lots of people forgot to take this branch on the interval $(0,1)$. Which means this part was forgotten.

(a) Sketch a graph of $f(x)$.

| $x$ | $f(x)$ |
|---|---|
| $-1$ | $-1$ |
| $0$ | $0$ |
| $1$ | $-1^2+1=0$ |
| $2$ | $-2^2+1=-3$ |
| $3$ | $-3^2+1=-8$ |




(b) What is $f(1)$?

$$f(1) = -1^2 + 1 \underset{\text{law 1}}{\overset{\text{negative}}{=}} (-1)\cdot 1^2 + 1 = -1+1 = \boxed{0}$$

(c) Does $\lim_{x\to 1} f(x)$ exist? If it does, find the limit. If not, explain why.

No, because looking @ the graph above:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = \boxed{1}$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} -x^2+1 = \boxed{0}$$

Since $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$, $\lim_{x\to 1} f(x)$ DNE.

4. Perform the given instruction. Remember to use the relevant laws/properties and **fully simplify**.

Not simplifying = lose points.

(a) Expand and simplify: $\dfrac{3(x+h)^2-1-(3x^2-1)}{h}$

common mistake:

$3(x+h)^2 \neq (3x+3h)^2$

because $3(x+h)^2 = 3\cdot(x+h)\cdot(x+h)$

You can only distribute the 3 to one factor of $(x+h)$.

3 multiplies into 3 terms.

Don't forget parenthesis.

$$\frac{3\underbrace{(x+h)}_{(A+B)}{}^2-1-(3x^2-1)}{h} \overset{\text{dist}}{=} \frac{3(x^2+2xh+h^2)\cancel{-1}-3x^2\cancel{+1}}{h}$$

$$\overset{\text{dist}}{=} \frac{\cancel{3x^2}+6xh+3h^2\cancel{-3x^2}}{h}$$

$$= \frac{6xh+3h^2}{h} \overset{\text{GCF}}{=} \frac{\cancel{h}(6x+3h)}{\cancel{h}} \underset{\text{law 5}}{\overset{\text{frac}}{=}} \boxed{6x+3h}$$

(b) $\boxed{\text{Expand:}}$ $(x^3+6)(2x+1)-(x^2+x-2)(3x^2)$

Convert to terms, no parenthesis.

$$(x^3+6)(2x+1)-(x^2+x-2)(3x^2) \overset{\text{dist}}{=} (x^3+6)2x+(x^3+6)\cdot 1-3x^4-3x^3+6x^2$$

$$\overset{\text{dist}}{=} 2x^4+12x+x^3+6-3x^4-3x^3+6x^2$$

$$= \boxed{-x^4-2x^2+6x^2+12x+6}$$

Common mistake: forgetting to distribute the factor of (-1)

(c) Rationalize the numerator: $\frac{\sqrt{x+h}-\sqrt{x}}{h}$

the only technique to square two terms.

$$(A - B)\cdot(A+B) = A^2 - B^2$$

NOT $(a+b)^2 = a^2 + b^2$

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h}+\sqrt{x}\right)}$$

frac law 5

$$= \boxed{\frac{1}{\sqrt{x+h}+\sqrt{x}}}$$

mistake #1

$$\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)^2 \neq \frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h^2}$$

because the numerator are terms and exponents don't interact with terms.

mistake #2

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\sqrt{x+h}+\sqrt{x}$$

forgot denominator $\sqrt{x+h}+\sqrt{x}$

you must pass the Vertical Line

5. Draw the **graph** of a function which satisfies the following:

   (a) $f(0) = 1$

   (b) $f(2) = 1$

   (c) $\lim_{x \to 0} f(x) = 1$

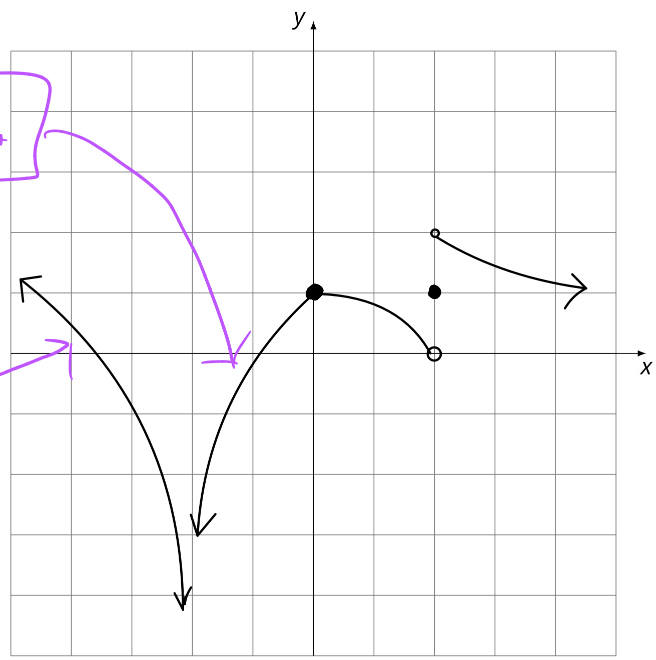
   (d) $\lim_{x \to 2^-} f(x) = 0$

   (e) $\lim_{x \to 2^+} f(x) = 2$

   (f) $\boxed{\lim_{x \to -2} f(x) = -\infty}$

Answers may vary

both $\lim_{x \to -2^-}$ and $\lim_{x \to -2^+}$ must agree.



6. Consider this limit:

$$\lim_{h\to 0}\frac{\dfrac{1}{3+h}-\dfrac{1}{3}}{h}$$

(a) Try using Limit Laws to find the limit. What ends up happening?

$$\lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h} \underset{(1),(2)}{\overset{\text{limit law } (4)}{=}} \frac{\dfrac{\lim_{h\to0} 1}{\lim_{h\to0} 3+\lim_{h\to0} h}-\lim_{h\to0}\left(\frac{1}{3}\right)}{\lim_{h\to0} h}$$

$$\underset{(6),(7)}{\overset{\text{limit law}}{=}} \frac{\frac{1}{3+0}-\frac{1}{3}}{0}$$

$$= \boxed{\frac{0}{0} \leftarrow \text{You end up with an indeterminate form of type } \frac{0}{0}.}$$

(b) Now find the actual limit.

The $\lim_{h\to 0}$ says you're looking to create a global factor of $h-0=\boxed{h}$ in the numerator. So, simplify the compound fraction.

$$\lim_{h\to 0}\frac{\frac{1}{(3+h)}-\frac{1}{3}}{h} = \lim_{h\to 0}\frac{\frac{3}{3}\cdot\frac{1}{(3+h)}-\frac{1}{3}\cdot\frac{3+h}{3+h}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{3}{3(3+h)}-\frac{3+h}{3(3+h)}}{h} \quad \text{frac law (1)}$$

$$= \lim_{h\to 0}\frac{\frac{3-(3+h)}{3(3+h)}}{h} \quad \text{frac law (3)}$$

$$= \lim_{h\to 0}\frac{\frac{3-3-h}{3(3+h)}}{h} \quad \text{dist law}$$

$$= \lim_{h\to 0}\frac{\frac{-h}{3(3+h)}}{h}$$

$$= \lim_{h\to 0}\frac{-h}{3(3+h)}\cdot\frac{1}{h} \quad \text{frac law (2)}$$

$$= \lim_{h\to 0}\frac{-\cancel{h}}{3\cdot\cancel{h}\cdot(3+h)} \quad \text{frac law (5)}$$

$$= \lim_{h\to 0}\frac{-1}{3(3+h)} \quad \text{now use limit laws}$$

$$= \frac{-1}{3(3+0)}$$

$$= \boxed{-\frac{1}{9}}$$

7. Use the **three-part definition of continuity** to prove the function

$$f(x) = \begin{cases} x(x-1) & x < 1 \\ 0 & x = 1 \\ \sqrt{x-1} & x > 1 \end{cases}$$

is continuous at the number $x = 1$.

Your answer needs to be as complete as my solution or points are lost.

① Show $\lim_{x \to 1} f(x)$ exists.

limit laws

we have: $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sqrt{x-1} = \sqrt{\lim_{x \to 1^+} x - \lim_{x \to 1^+} 1} = \sqrt{1-1} = 0$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} [x(x-1)] = \left[\lim_{x \to 1^-} x\right]\left[\lim_{x \to 1^-} x - \lim_{x \to 1^-} 1\right]$

limit laws

$= 1 \cdot (1-1)$

$= 0$

Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$, we conclude $\lim_{x \to 1} f(x)$ exists and is equal to 0.

② Show $f(1)$ is defined.

$f(1) = 0$ ✓

③ Show $\lim_{x \to 1} f(x) = f(1)$

from parts ① and ②, $\lim_{x \to 1} f(x) = 0$ and $f(1) = 0$.

∴, this condition is satisfied.

By the definition of continuity $f(x)$ is continuous at $x = 1$.

8. Answer the following:

(a) For a function $f(x)$, what is the **limit definition** of the derivative?

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(b) Suppose

$$f(x) = 2x^2 - 1$$

**Using the limit definition of the derivative**, find $f'(x)$.

Not using the limit definition (i.e. using shortcuts) = 0 points.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

subtracting $\geq 2$ terms

$$= \lim_{h \to 0} \frac{2(x+h)^2 - 1 - (2x^2 - 1)}{h}$$

2 multiplies into three terms.

$$= \lim_{h \to 0} \frac{2(x^2 + 2xh + h^2) - 1 - 2x^2 + 1}{h}$$

$$= \lim_{h \to 0} \frac{\boxed{2x^2} + 4xh + 2h^2 \boxed{-1} \boxed{-2x^2} \boxed{+1}}{h}$$

$$= \lim_{h \to 0} \frac{4xh + 2h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(4x + 2h)}{\cancel{h}}$$

$$= \lim_{h \to 0} 4x + 2h$$

$$= 4x + 2 \cdot 0$$

$$= \boxed{4x}$$

9. Find the derivative of the following functions.

(a) $f(x) = 534534532$

$$f'(x) = \frac{d}{dx}[534534532] = \boxed{0}$$

(b) $g(t) = -t$

exponent is 1

$$g'(t) = \frac{d}{dt}[-t] = -\frac{d}{dt}[t]$$

$$= -1 \cdot t^{1-1}$$

$$= -1 \cdot t^0$$

$$= -1 \cdot 1$$

$$= \boxed{-1}$$

(c) $f(x) = 4x^3 - 2x^2 + x - 5$

$$f'(x) = \frac{d}{dx}[4x^3 - 2x^2 + x - 5]$$

$$= 4\frac{d}{dx}[x^3] - 2\frac{d}{dx}[x^2] + \frac{d}{dx}[x] - \frac{d}{dx}[5]$$

$$= 4 \cdot 3 \cdot x^{3-1} - 2 \cdot 2 \cdot x^{2-1} + 1 \cdot x^{1-1} - 0 = \boxed{12x^2 - 4x - 1}$$

(d) $g(\theta) = \theta \cdot \sqrt{\theta} \cdot \theta^3 \cdot \theta^4$ ← simplify.

Know your exponent laws
Lecture Note II.

$$= \theta \cdot \theta^{\frac{1}{2}} \cdot \theta^3 \cdot \theta^4$$

$$= \theta^{1 + \frac{1}{2} + 3 + 4}$$

$$= \theta^{8 + \frac{1}{2}}$$

$$= \theta^{\frac{17}{2}}$$

$$g'(\theta) = \frac{d}{d\theta}\left[\theta^{\frac{17}{2}}\right]$$

$$= \frac{17}{2}\theta^{\frac{17}{2} - 1}$$

$$= \boxed{\frac{17}{2}\theta^{\frac{15}{2}}}$$